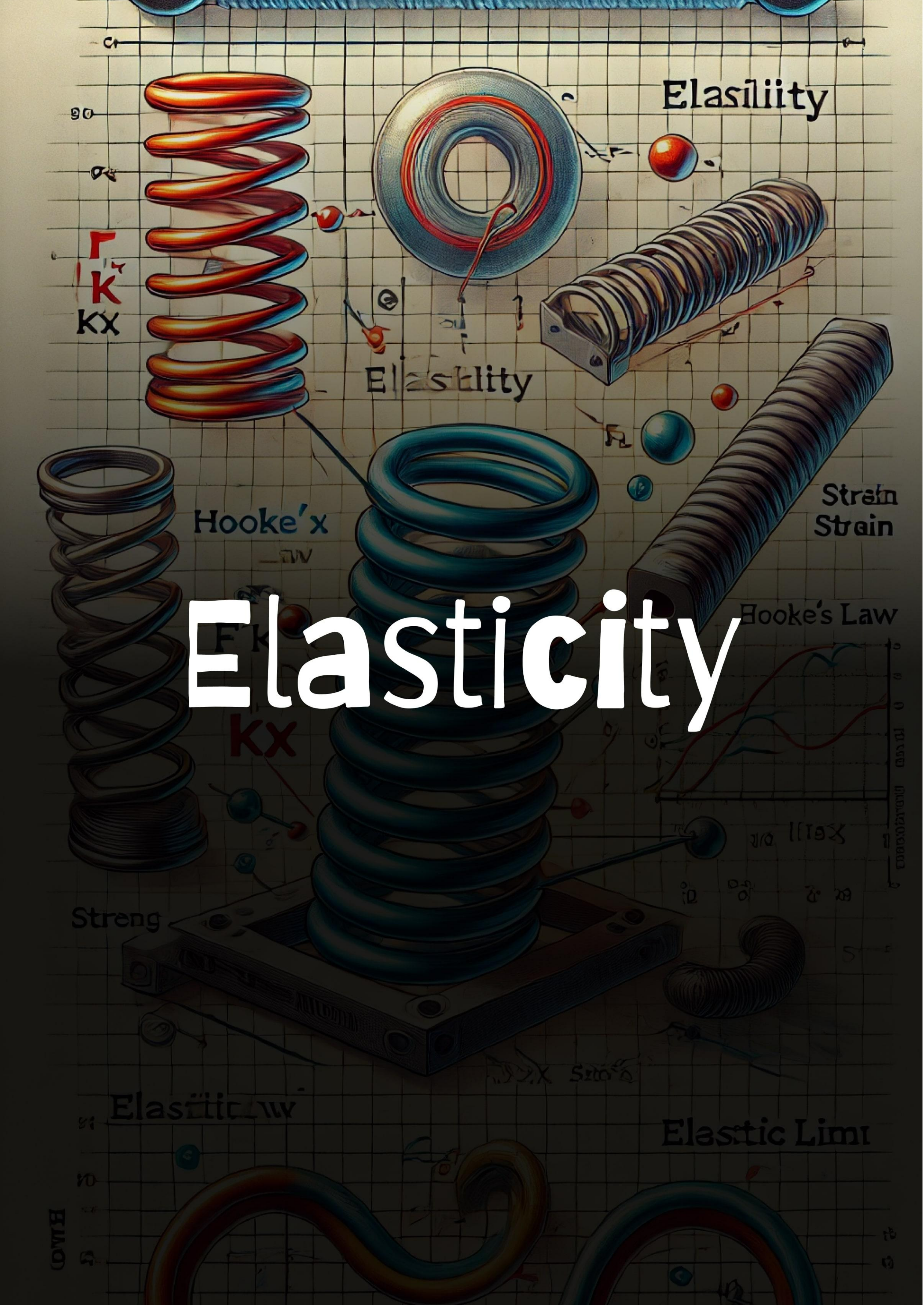
Elasticity



Elasiliity
KX
Hooke'x
Strain
Hooke's Law
Streng
Elastic Limi

Elasticity

LONGITUDINAL STRESS (σ)

If F be the tension along the length of the rod whose cross sectional area is A, then we define longitudinal stress as

$$\boxed{\sigma = \frac{F}{A}}$$

There are two types of stress:
(i) Engineering stress
(ii) True stress

STRAIN (ϵ)

If an element of original length dx elongates by a length dy due to longitudinal stress, then we define local strain as

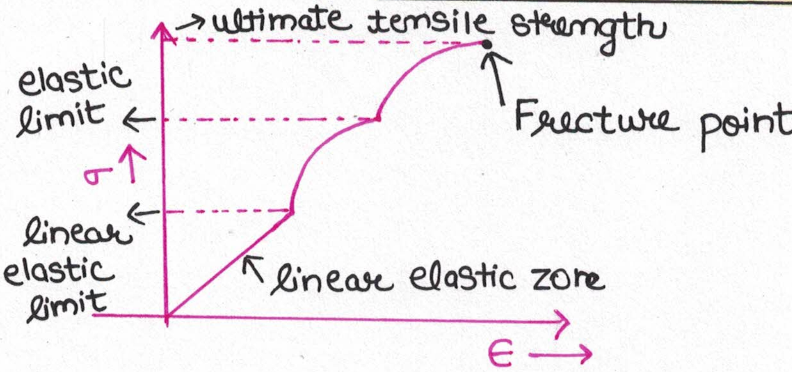
$$\boxed{\epsilon = \frac{dy}{dx}}$$

If all the elements of a rod have same strain then we can also write

$$\epsilon = \frac{\Delta l}{l}$$

where Δl is change in length and l is original length.

STRESS Vs STRAIN (GRAPH)



YOUNG'S MODULUS (Y)

The ratio of stress to strain in the linear elastic limit is called young's modulus.

$$\boxed{Y = \frac{\sigma}{\epsilon}}$$

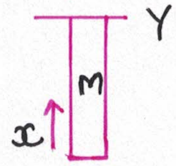
$$\boxed{Y = \frac{Fl}{A\Delta l}}$$

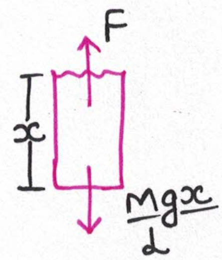
Que.) A rod of mass M & length L (unstretched) hang from a ceiling. Find

(i) Tension as a function of distance from the bottom

(ii) σ & ϵ as a function of distance x

(iii) Total elongation of rod.

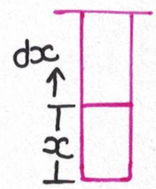




(i) $F = \dfrac{Mgx}{L}$

(ii) $\sigma = \dfrac{Mgx}{LA}$

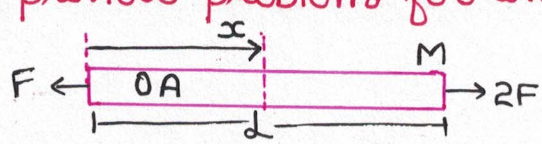
$$\epsilon = \frac{Mgx}{LAY} = \frac{dy}{dx}$$



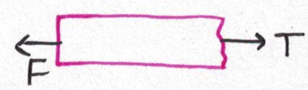
(iii) $\displaystyle\int_0^{\Delta L} dy = \frac{Mg}{LAY}\int_0^{L} x \cdot dx$

$$\Delta L = \frac{MgL}{2AY}$$

Que.) Repeat the previous problem for the shown situation.



$$T - F = \frac{Mx}{L} \cdot \frac{F}{M}$$



$$T = F\left(\frac{x}{L} + 1\right)$$

$$\sigma = \frac{F}{LA}(x + L)$$

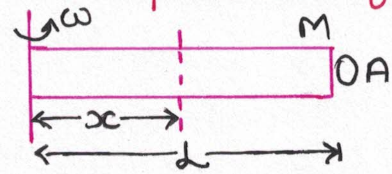
$$\epsilon = \frac{F}{LAY}(L + x) = \frac{dy}{dx}$$

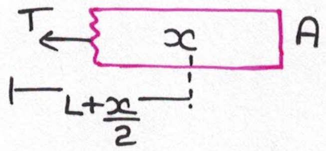
$$\int_0^{\Delta L} dy = \int_0^{L} \frac{F}{LAY}(L + x) \cdot dx$$

$$\Delta L = \frac{F}{LAY}\left[Lx + \frac{x^2}{2}\right]_0^{2}$$

$$\Delta L = \frac{3}{2}\frac{FL}{AY}$$

Que.) Repeat the previous problem for the shown situation.





$$T = \frac{\omega^2 (L+x)}{2} \times \frac{M(L-x)}{L}$$

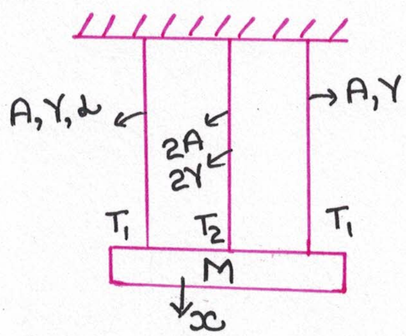
$$T = \frac{M\omega^2}{2L}(L^2 - x^2)$$

$$\int_0^{\Delta L} dy = \frac{M\omega^2}{2LAY}\int_0^{L}(L^2-x^2).dx$$

$$\Delta L = \frac{M\omega^2}{2LAY}\left(L^3 - \frac{L^3}{3}\right)$$

$$= \frac{M\omega^2 L^2}{3AY}$$

Que.) A block of mass M hangs by 3 wires, two of them being identical as shown. Find the tensions T_1 and T_2 and also the elongation x.



$$Y = \frac{FL}{A\,\Delta L}$$

$$F = \frac{AY\Delta L}{L}$$

$$F = \frac{AYx}{L}$$

$$T_1 = \frac{AYx}{L}$$

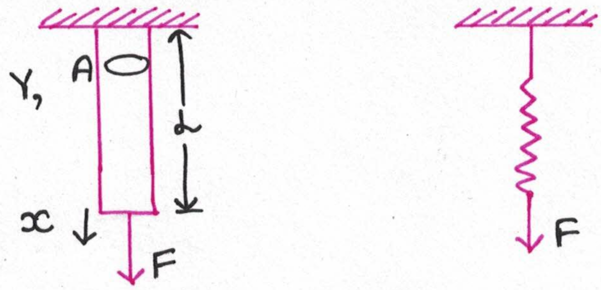
$$T_2 = \frac{4YxA}{L}$$

$$\frac{2AYx}{L} + \frac{4AYx}{L} = Mg$$

$$x = \frac{MgL}{6AY}$$

$$T_1 = \frac{Mg}{6}, \quad T_2 = \frac{2Mg}{3}$$

SPRING EQUIVALENT OF A MASSLESS ELASTIC ROD

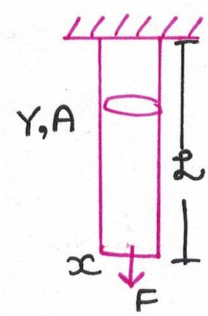


$$Y = \frac{Fl}{Ax}$$

$$F = \frac{YAx}{l} = K_{eff} \cdot x$$

$$\boxed{K_{eff} = \frac{YA}{l}}$$

ELASTIC ENERGY DENSITY IN A ROD



$$u = \frac{1}{2} \times \left(\frac{YA}{l}\right) x^2$$

$$V = Al \quad \text{(volume)}$$

$$u = \frac{u}{V} = \frac{1}{2}\frac{Y\cancel{A}}{l} \cdot x^2 \times \frac{1}{\cancel{A}l}$$

$$= \frac{1}{2} Y \left(\frac{x}{l}\right)^2$$

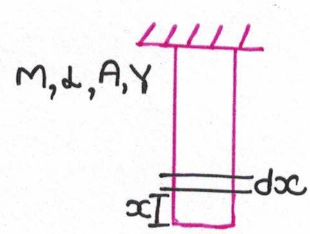
$$u = \frac{1}{2} \times Y \times \epsilon^2 \qquad (u \rightarrow \text{energy per unit volume})$$

$$Y = \frac{\sigma}{\epsilon}$$

$$u = \frac{1}{2}\sigma\epsilon$$

$$u = \frac{1}{2}\frac{\sigma^2}{Y}$$

Que.) Find elastic energy stored in the rod due to elongation under its own weight.



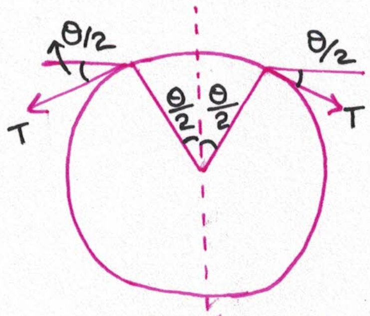
$$\epsilon = \frac{Mgx}{lAY}$$

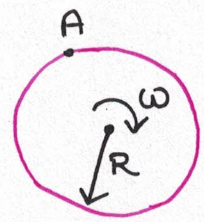
$$du = \frac{1}{2} \times Y \times \left(\frac{Mgx}{lAY}\right)^2$$

$$\int_0^u du = \frac{M^2g^2}{2l^2AY}\int_0^l x^2 \cdot dx$$

$$u = \frac{M^2g^2l}{6AY}$$

Que.) Find (i) The tension in the ring as a function of ω & (ii) Maximum permissible ω so that ring does not break.





$$2T\sin\left(\frac{\theta}{2}\right) = \rho(AR\theta)\,\omega^2 R$$

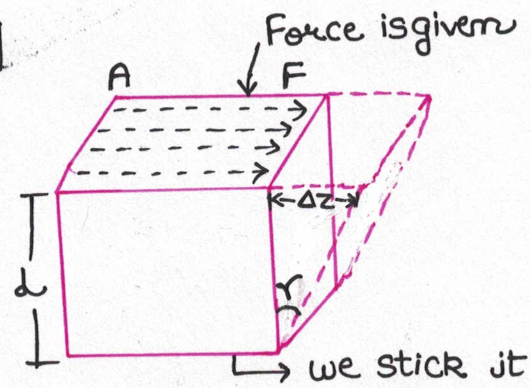
$$T\cancel{\theta} = \rho A\omega^2R^2\cancel{\theta}$$

$$T = \rho A\omega^2 R$$

$$\sigma = \frac{T}{A} = \rho\omega^2R^2 < \sigma_0$$

$$\omega < \sqrt{\frac{\sigma_0}{\rho R^2}}$$

SHEAR STRESS



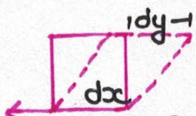
If a force F acts parallel to a surface area A, then we define the shear stress as

$$\left(\sigma_s = \frac{F}{A}\right)$$

SHEAR STRAIN

Let the upper edge of the shown block get displaced through a distance Δl more, then we define shear strain as

$$\epsilon_s = \frac{\Delta l}{l} = \tan\gamma$$



$$\epsilon_s = \frac{dy}{dx}$$

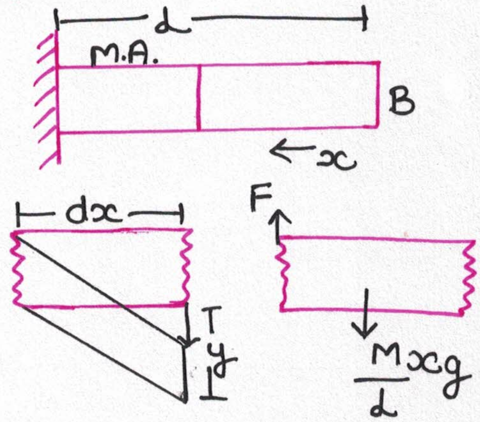
where l is the vertical thickness of the block.

SHEAR MODULUS

The ratio of shear stress to shear strain is called shear modulus.

$$\boxed{\eta = \frac{\sigma_s}{\epsilon_s}}$$

Que.) A candylever beam comes out of the wall as shown in figure. Find
(i) shear stress as a function of x.
(ii) shear strain as a function of x.
(iii) Amount by which end B comes down.



$$F = \frac{Mxg}{L}$$

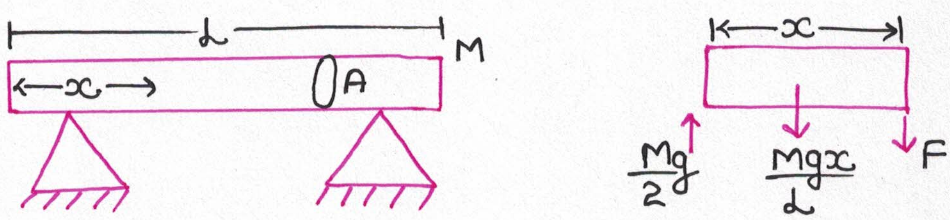
$$\sigma_s = \frac{F}{A} = \frac{Mgx}{LA}$$

$$\epsilon_s = \frac{\sigma}{\eta} = \frac{Mg.x}{LA\eta} = \frac{dy}{dx}$$

$$\int_0^{\Delta L} dy = \frac{Mg}{LA\eta}\int_0^{L} x.dx$$

$$\Delta L = \frac{MgL}{LA\eta}$$

Que.) Find shear strain and shear stress.



$$F = \frac{Mg}{2} - \frac{Mg.x}{L}$$

$$F = Mg\left[\frac{1}{2} - \frac{x}{L}\right]$$

$$\sigma_s = \frac{F}{A} = \frac{Mg}{A}\left[\frac{1}{2} - \frac{x}{L}\right]$$

Que.) Find the max. displacement in the previous que.

$$\epsilon = \frac{\sigma_s}{\eta}$$

$$\epsilon = \frac{Mg}{2A\eta}\left[1 - \frac{2x}{L}\right] = \frac{dy}{dx}$$

$$\int_0^{\Delta L} dy = \left[\frac{Mgx}{2A\eta} - \frac{x^2 Mg}{2LA\eta}\right]_0^{L/2}$$

$$\Delta L = \frac{MgL}{4A\eta} - \frac{LMg}{A\eta\, 8} = \frac{MgL}{8A\eta}$$

BULK STRESS

Let a substance be surrounded by an atmosphere of pressure P and let dP be the change in pressure, then the bulk stress is defined as

$$\sigma_B = dP$$

BULK STRAIN

Let dv be change in volume of a substance having original volume v. Then we define bulk strain as

$$\epsilon_B = \frac{dv}{v}$$

BULK MODULUS

The magnitude of ratio of bulk stress to bulk strain is called bulk modulus.

$$B = \frac{-dP}{dv/v} = \frac{-\sigma_B}{\epsilon_B}$$

Que.) Find the bulk modulus of an ideal gas for an adiabatic process.

$$Pv^\gamma = C$$

$$\ln P + \gamma \ln v = \ln C$$

$$\frac{1}{P}\frac{dP}{dv} + \frac{\gamma}{v} = 0$$

$$\frac{-dP}{dv/v} = \gamma P$$

★ For laboratory purposes, we can write,

$$B = \frac{-\Delta P}{(\Delta v/v)}$$

BULK MODULUS IN TERMS OF DENSITY

$$\rho = \frac{m}{v}$$

$$\ln \rho = \ln m - \ln v$$

$$\frac{1}{\rho}\frac{d\rho}{dv} = -\frac{1}{v} \qquad \text{(w.r.t. } v\text{)}$$

$$\frac{-dv}{v} = \frac{d\rho}{\rho}$$

$$\boxed{B = \frac{dP}{(d\rho/\rho)}}$$

Que.) The density on the surface of well is ρ_0, and bulk modulus is B_0 (assume constant). Find relation b/w depth & density.

$$P = P_0 + \rho g y$$

$$dP = \rho g\, dy$$

$$B_0 = \frac{dP}{(d\rho/\rho)}$$

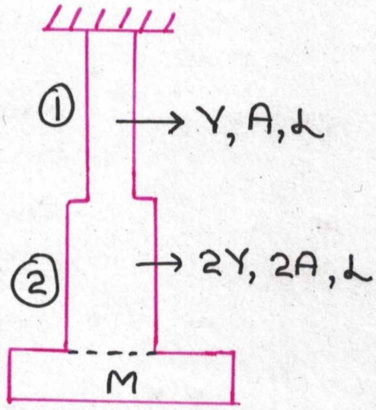
$$\frac{B_0}{\rho} = \frac{\rho g\, dy}{d\rho}$$

$$\int_{\rho_0}^{\rho} \frac{B_0}{\rho^2 g} \cdot d\rho = \int_0^d dy$$

$$\left[\frac{-B_0}{\rho g}\right]_{\rho_0}^{\rho} = [y]_0^d$$

$$\frac{gy}{B_0} = -\frac{1}{\rho} + \frac{1}{\rho_0}$$

Que.) Find the individual elongations and total elongation. Also find Keff. of the combination.



$$T = Mg$$

$$\sigma_T = \frac{Mg}{A}$$

$$\epsilon_1 = \frac{Mg}{AY} = \frac{x_1}{L}$$

$$x_1 = \frac{MgL}{AY}$$

Similarly, $x_2 = \dfrac{MgL}{4AY}$

$$x = x_1 + x_2 = \frac{5MgL}{4AY}$$

$$K_{eff} = \frac{Mg}{x}$$

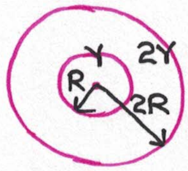
$$= \frac{4AY}{5L}$$

COMPRESSIBILITY

The reciprocal of ~~block~~ bulk modulus is called compressibility.

$$K = \frac{1}{B} = \frac{-\epsilon_B}{\sigma_B}$$

Que.) Find the elongation in the cable and elastic energy stored in the cable.



$$K_1 = \frac{Y\pi R^2}{L}$$

$$\therefore K_2 = \frac{2Y \cdot 3\pi R^2}{L} = \frac{6Y\pi R^2}{L}$$

$$K_{eff} = K_1 + K_2$$

$$= \frac{7Y\pi R^2}{L} = \frac{Mg}{x}$$

$$x = \frac{MgL}{7Y\pi R^2}$$

$$U = \frac{1}{2} K_{eff} \cdot x^2$$

$$U = \frac{1}{2}\left(\frac{Mg}{x}\right)x^2$$

$$U = \frac{Mgx}{2} = \frac{M^2g^2L}{14\,YAR^2}$$